Online Gradient Based Optimisation

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AI801 Lecture on 27.04.2026

University of
Southern Denmark

Special Topics Outline

- What more can you do with Gradient Descent, especially in AI/ML?
- Three main topics:
  - Online Gradient Descent (Today)
  - Distributed Gradient Descent (May 4th)
  - Communication-Efficient Gradient Descent (May 11th)
- Cornerstone of modern machine learning (incl. deep learning, LLMs).
- Lecture slides, assignment and reading materials will be posted to the course web page (Unit 7).
- Main reference material for today is Elad Hazan's book on Online Convex Optimization.

Motivation


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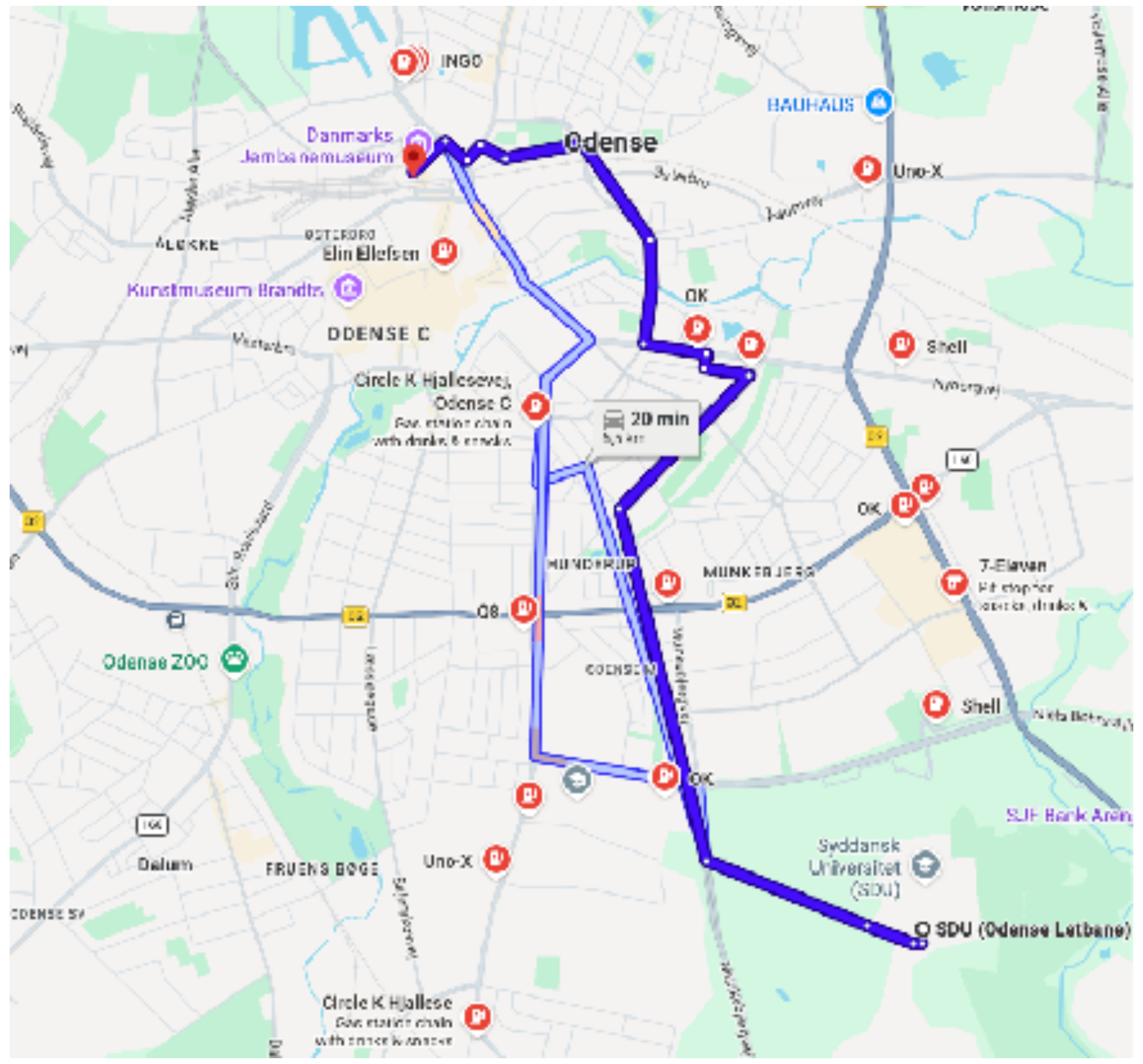
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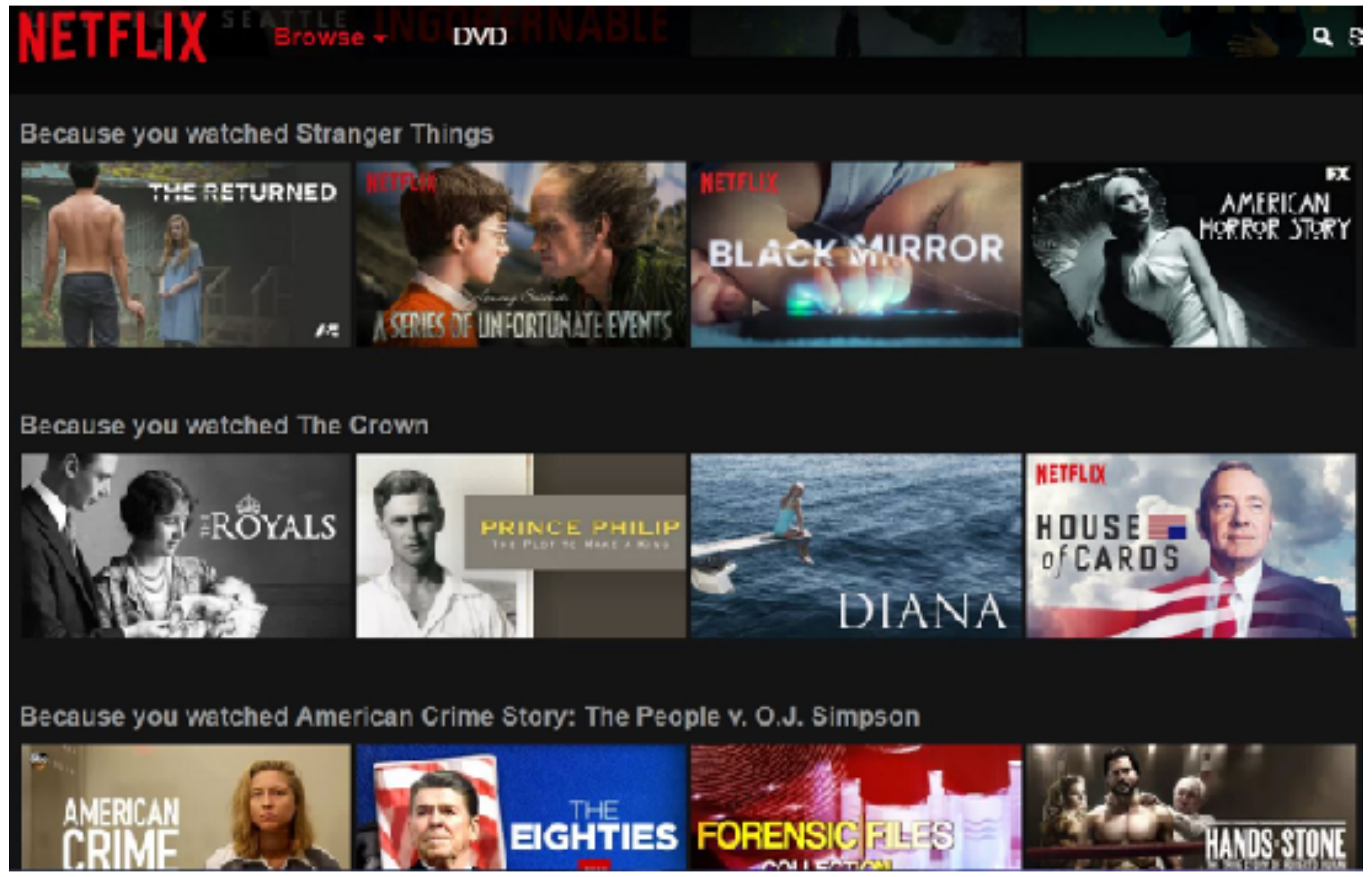
NETFLIX
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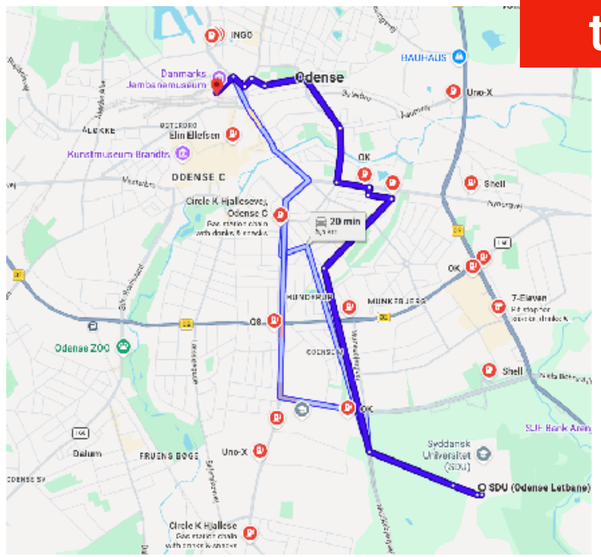
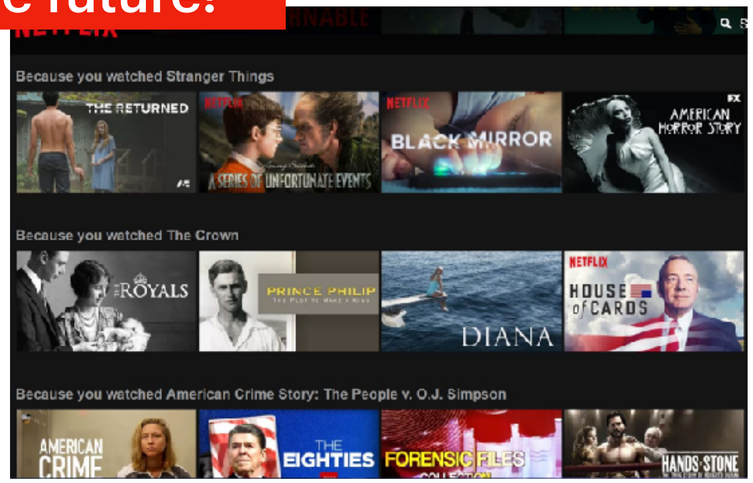
THE RETURNED
BLACK MIRROR
AMERICAN HORROR STORY
Because you watched The Crown
ROYALS
PRINCE PHILIP
DIANA
HOUSE of CARDS
Because you watched American Crime Story: The People v. O.J. Simpson
AMERICAN CRIME
THE EIGHTIES
FORENSIC FILES

Motivation

Take decisions now without the knowledge of the future!



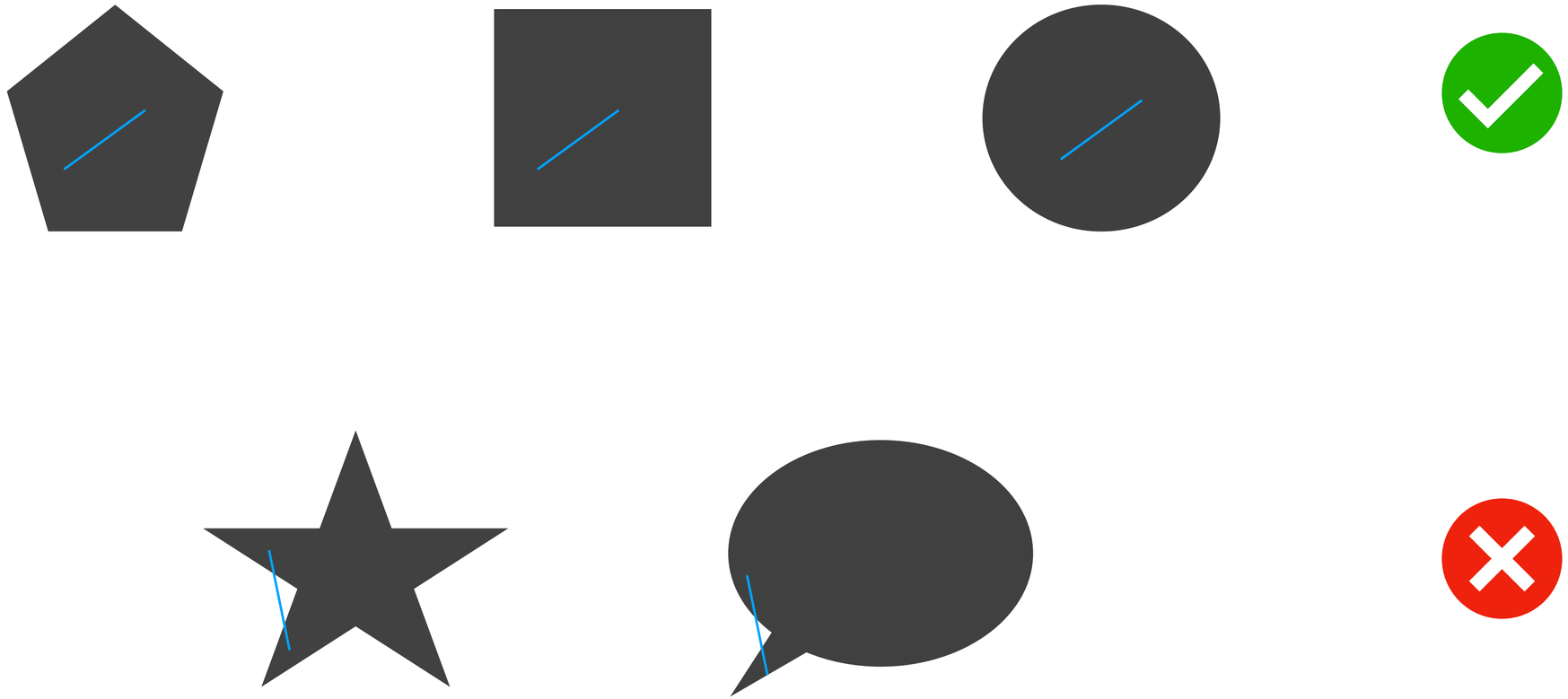


Online Optimization

- **Online** = We (player/agent) take decisions iteratively and obtain a *loss* or a *gradient* at each step, without knowing what is coming ahead!
- How do we “optimize” if we do not know *what* to optimize?
- Who decides the losses?
- Particularly, we will only be concerned with Online *convex* optimization (OCO).
- *When* can we say something meaningful about what the player can do in such a setting?

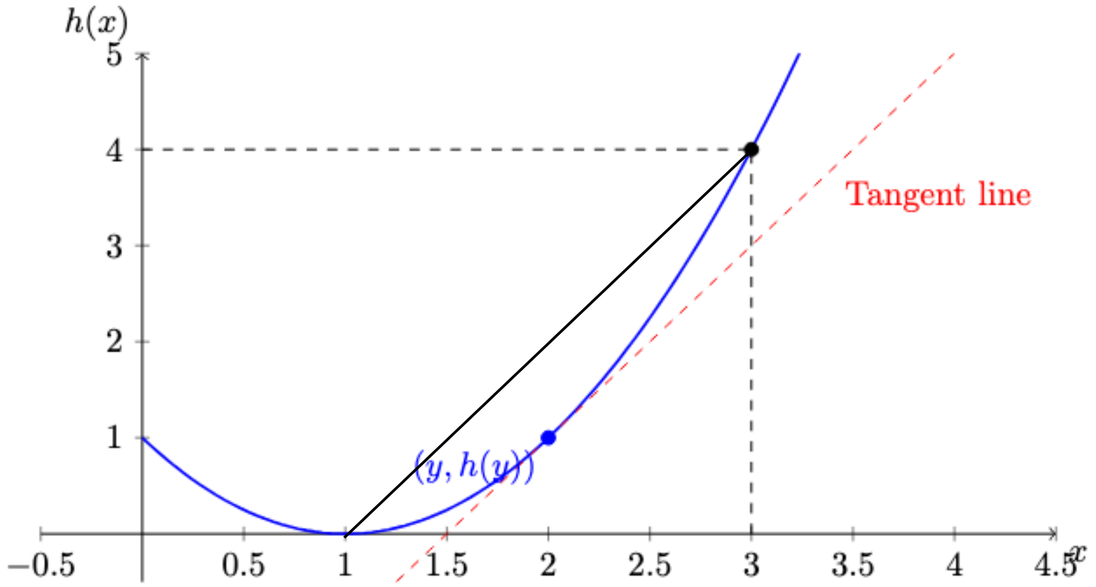
Quick Recap

Convex Sets



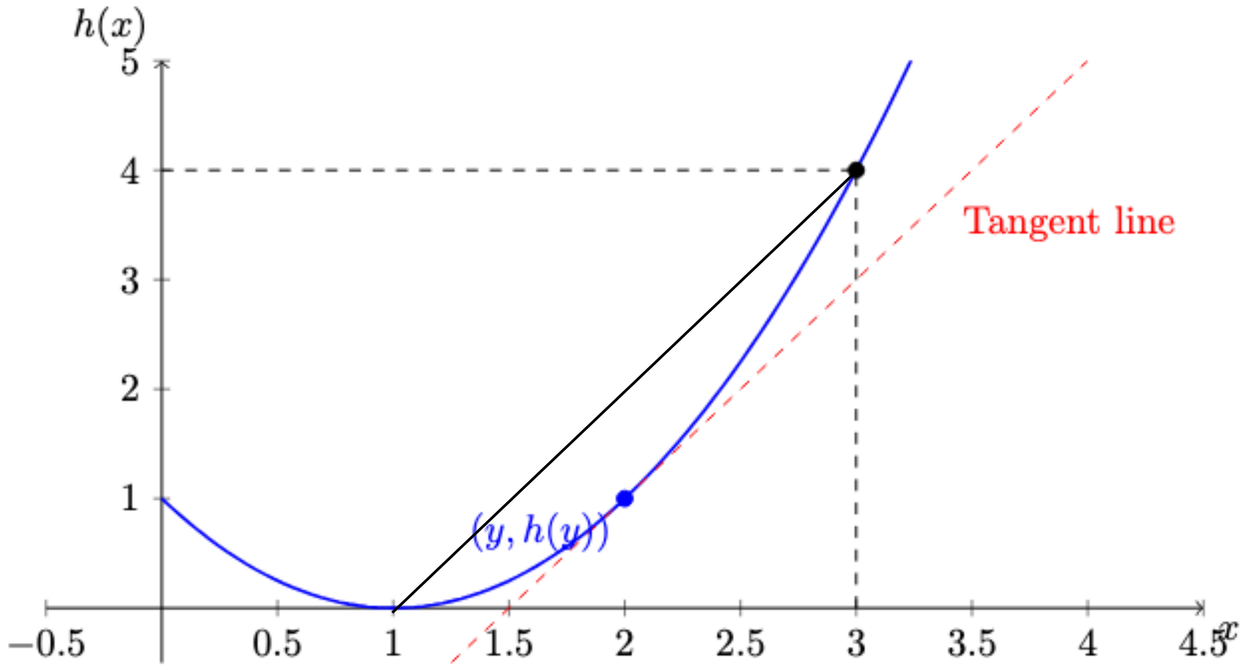
A set $\mathcal{X} \subseteq \mathbb{R}^d$ is convex if and only if for all $x, y \in \mathcal{X}$, we have: $\lambda x + (1-\lambda)y \in \mathcal{X}$, for all $\lambda \in [0,1]$

Convex Functions



A function $h : \mathcal{X} \to \mathbb{R}$ is convex, if and only if $h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y)$, for all $x, y \in \mathcal{X}$ and for all $\lambda \in [0,1]$

First-Order Conditions



A differentiable function $h : \mathcal{X} \to \mathbb{R}$ is convex, if and only if $h(x) \geq h(y) + \nabla h(y)^\top (x - y)$, for all $x, y \in \mathcal{X}$.

Smoothness and Strong Convexity

A differentiable function $h : \mathcal{X} \to \mathbb{R}$

is α-**strongly** convex,

if there is an $\alpha > 0$, such that:

$$h(x) \geq h(y) + \nabla h(y)^\top (x - y) + \frac{\alpha}{2}\|x - y\|_2^2,$$

for all $x, y \in \mathcal{X}$.

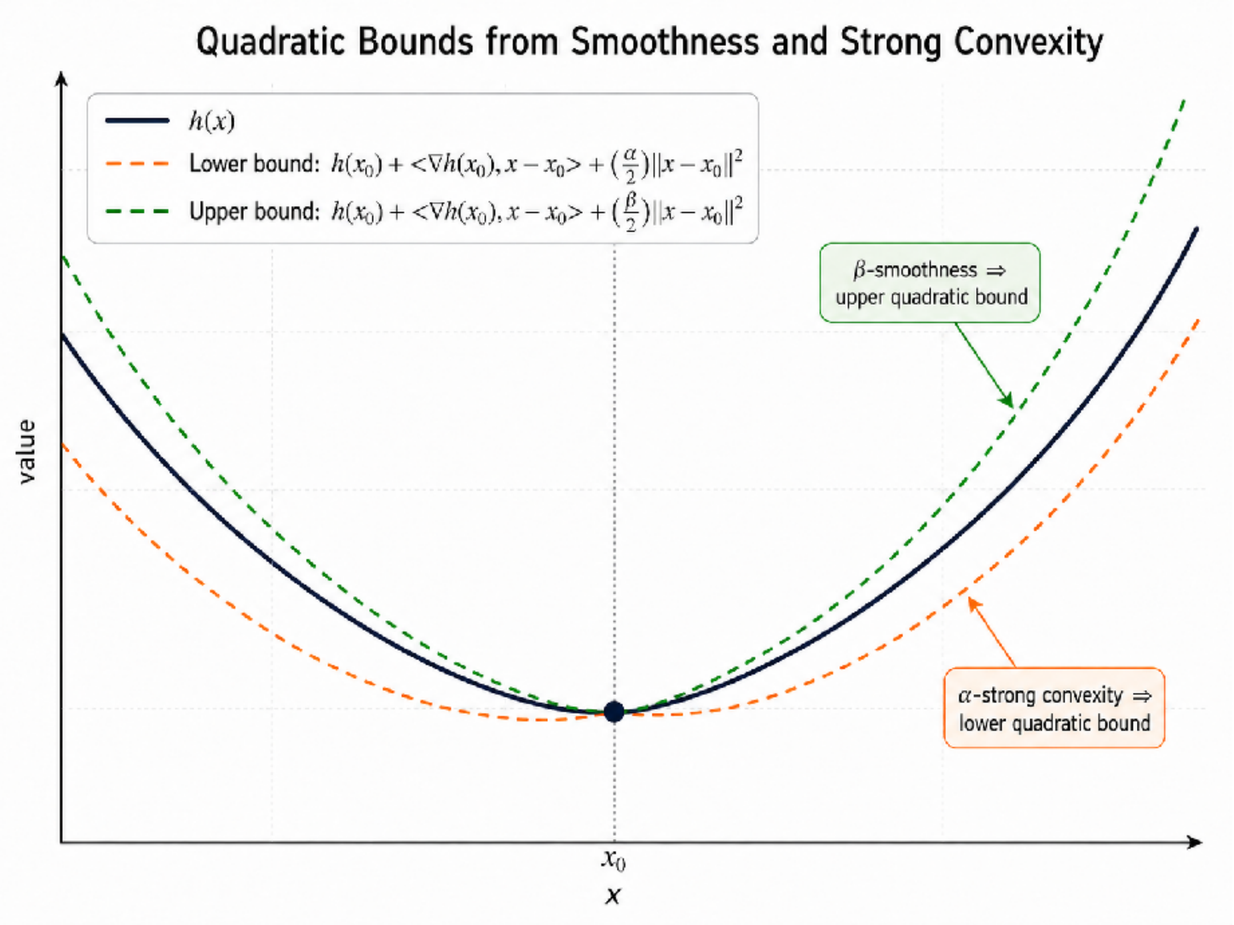
A differentiable function $h : \mathcal{X} \to \mathbb{R}$

is β-**smooth**

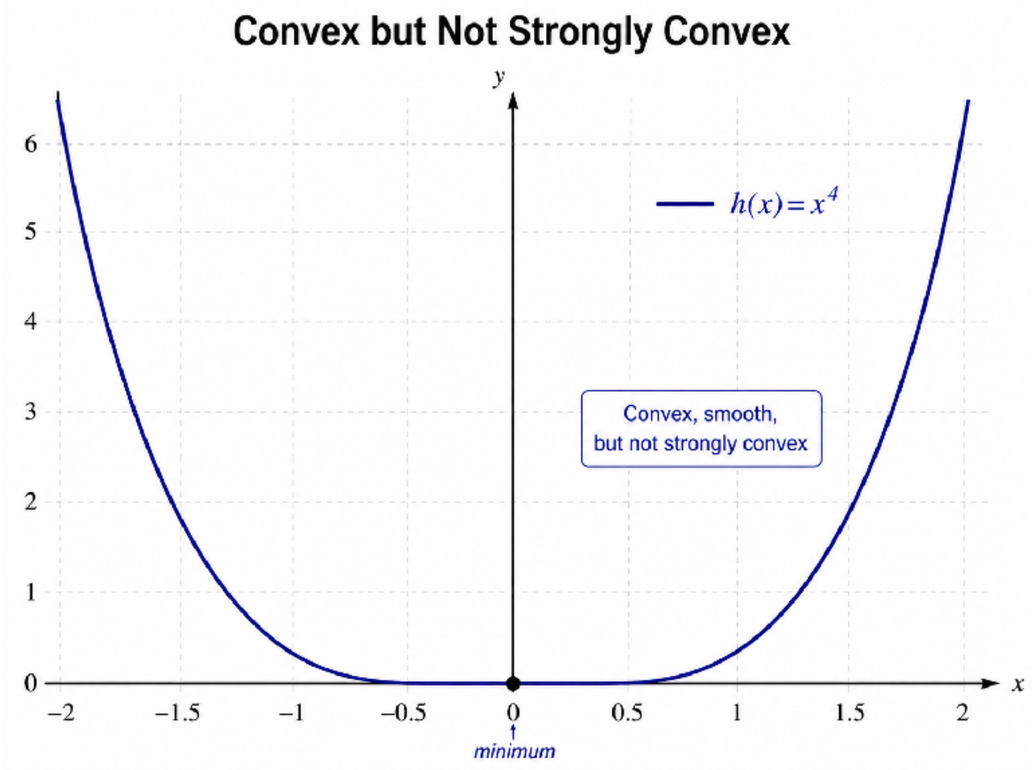
if there is an $\beta > 0$, such that:

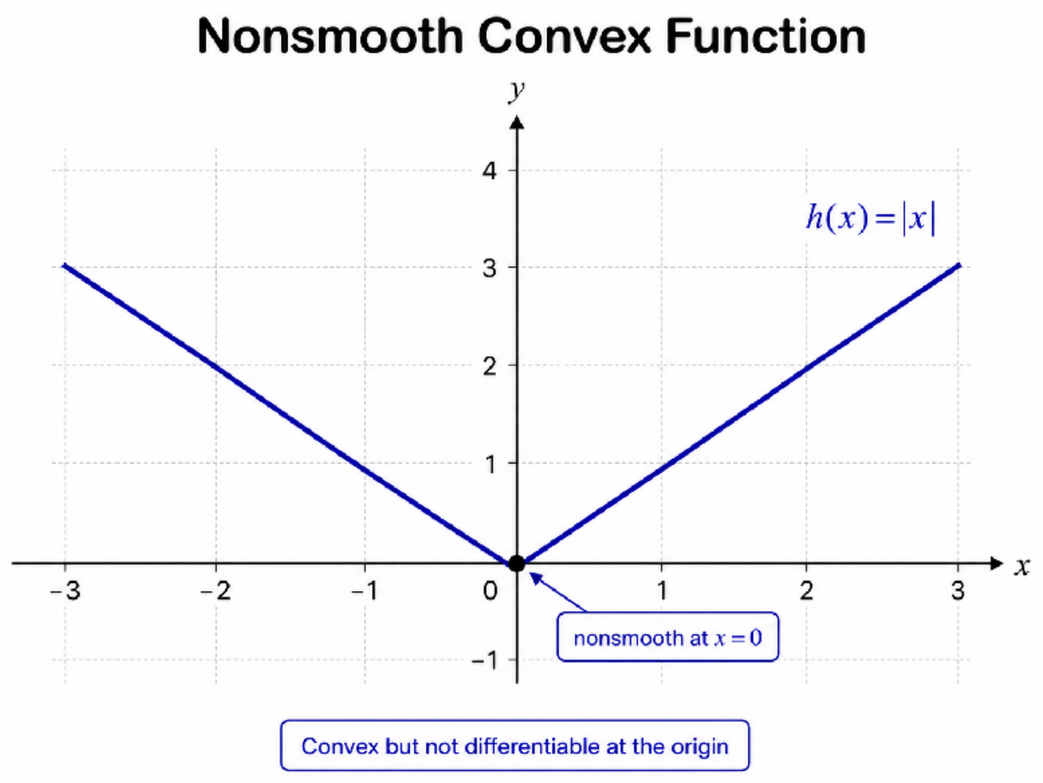
$$h(x) \leq h(y) + \nabla h(y)^\top (x - y) + \frac{\beta}{2}\|x - y\|_2^2,$$

for all $x, y \in \mathcal{X}$.

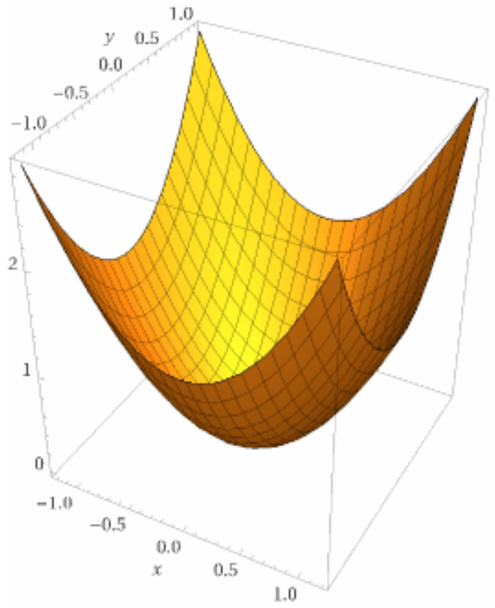


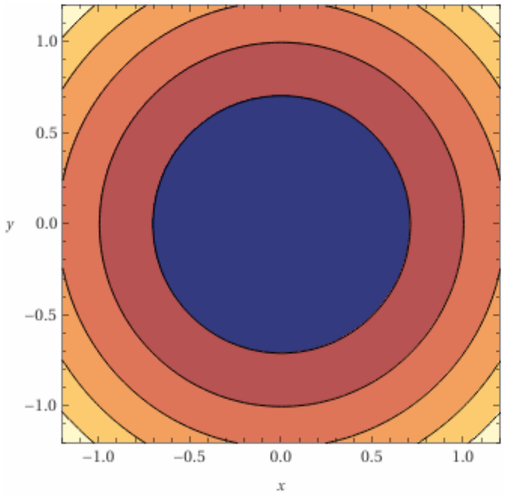
Other Convex Functions





Second-Order Conditions





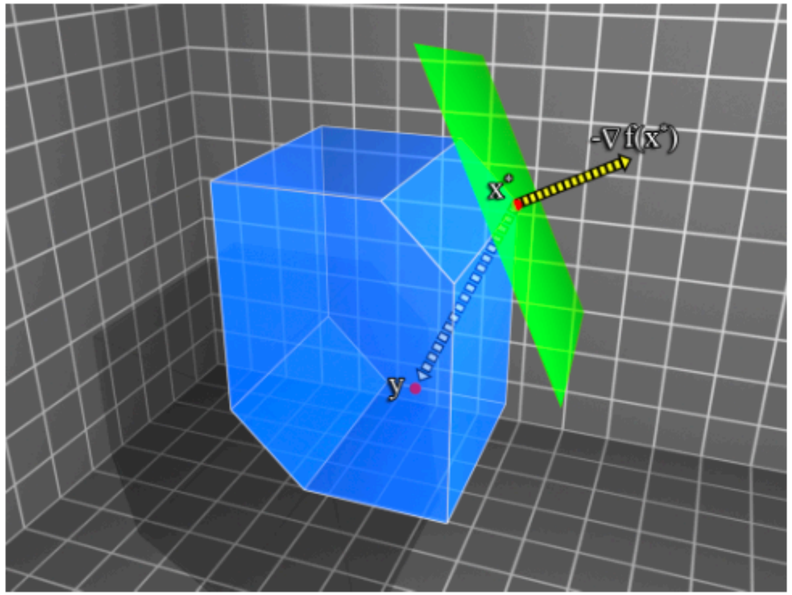
A twice differentiable function $h : \mathcal{X} \to \mathbb{R}$ is α-**strongly** convex, if there is an $\alpha > 0$, such that: $\nabla^2 h(x) \geq \alpha I$, for all $x \in \mathcal{X}$.

A twice differentiable function $h : \mathcal{X} \to \mathbb{R}$ is β-**smooth**, if there is an $\beta > 0$, such that: $\nabla^2 h(x) \leq \beta I$, for all $x \in \mathcal{X}$.

Key Facts about Optimality

- Convex functions over a compact (closed and bounded) domain always admit a global minimizer (may not be unique).
- All local minima are global minima!
- **Strongly** convex functions have a unique global minima.
- We will only consider the case when $\mathcal{X}$ is compact (e.g., polytopes, simplex, (closed) boxes etc.).

Optimality Conditions



A point $x^* \in \mathcal{X}$, is optimal for h if and only if: $\nabla h(x^*)^\top (y - x^*) \geq 0$ for all $y \in \mathcal{X}$.

Why?

Gradient Descent

Initialize with x_0

For $t = 0$ to $T - 1$ do,

$$x_{t+1} = x_t - \eta_t \nabla h(x_t)$$

Return $\bar{x}_T = \text{argmin}_{t=0,1,\ldots,T-1}\{h(x_t)\}$

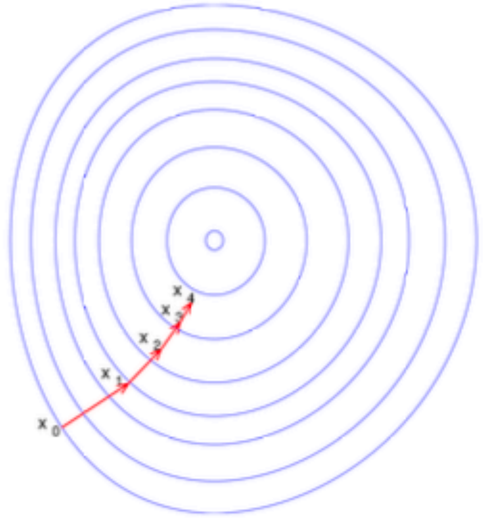


Figure 2.3: Iterates of the GD algorithm

Here we return the **best-iterate**, sometimes, we might consider,

the **last-iterate** x_T or an **average iterate** $\frac{1}{T}\sum_{t=0}^{T-1} x_t$

Convergence Rates of GD

	General	Strongly convex	Smooth	Well-conditioned
GD	$\frac{1}{\sqrt{T}}$	$\frac{1}{\alpha T}$	$\frac{\beta}{T}$	$\exp\left(-\frac{\alpha}{\beta}T\right)$

GD converges to a minimizer, i.e., $h(\bar{x}_T) - h(x^*) \leq O(r(T))$, under an appropriate choice of the sequence of step-sizes $\{\eta_t\}$.

Projected Gradient Descent

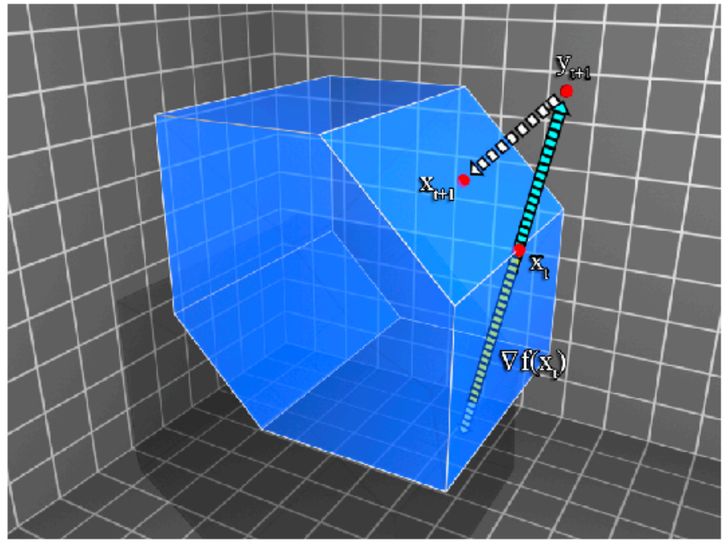
Initialize with $x_0 \in \mathcal{X}$

For $t = 0$ to $T - 1$ do,

$$y_{t+1} = x_t - \eta_t \nabla h(x_t)$$

$$x_{t+1} = \Pi_{\mathcal{X}}(y_{t+1})$$

Return $\bar{x}_T = \text{argmin}_{t=0,1,\ldots,T-1}\{h(x_t)\}$



PGD converges to a minimizer, i.e., $h(\bar{x}_T) - h(x^*) \leq O(r(T))$, with effectively the same rates (wrt dependence on T) and appropriate selection of $\{\eta_t\}$.

Online Convex Optimization

Setup

- The game proceeds in rounds $t = 0,1,\ldots,T$.
- At each iteration t, Player plays $x_t \in \mathcal{X}$ and gets $h_t(x_t)$ as feedback or loss.
- Player does not know future losses!
- We will assume that $\{h_t\}$ are a sequence of loss functions that are in some convex fixed function class (e.g., β-smooth, α-strongly convex etc.).
- But what is the right “optimizer” for the online setting?

Regret Minimization

Regret $\longrightarrow$ $$R(T) = \underbrace{\sum_{t=0}^{T-1} h_t(x_t)}_{\text{Total loss actually obtained during the T rounds.}} - \underbrace{\min_{x \in \mathcal{X}} \sum_{t=0}^{T-1} h_t(x)}_{\text{Total loss of the best fixed point (in hindsight)}}$$

Player's goal is to use $\{x_t\}_{t=0}^{T-1}$ such that: $\lim_{T \to \infty} \frac{R(T)}{T} = 0$

No-regret Algorithm: Update rules for x_t, that satisfy $R(T) = o(T)$.

Online Gradient Descent

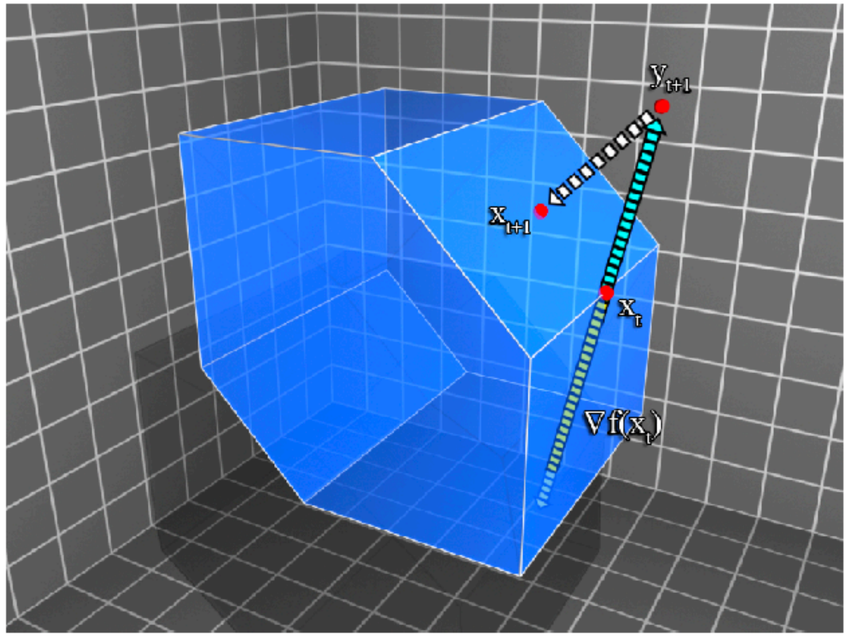
Initialize with $x_0 \in \mathcal{X}$

For $t = 0$ to $T - 1$ do,

$$y_{t+1} = x_t - \eta_t \nabla h_t(x_t)$$

$$x_{t+1} = \Pi_{\mathcal{X}}(y_{t+1})$$

Return x_T



OGD satisfies $R_T = o(T)$, with appropriate selection of $\{\eta_t\}$ and assuming uniformly bounded norm of gradients.

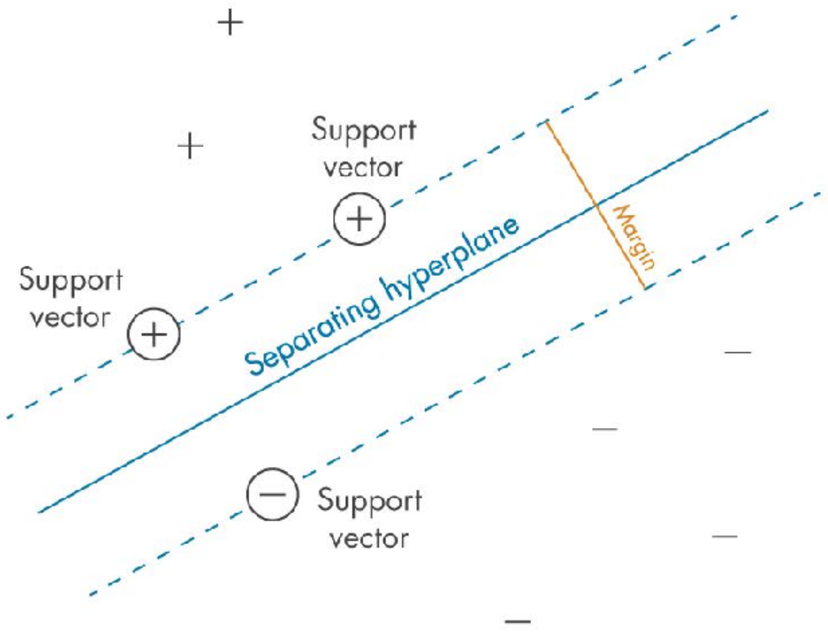
Regret of OGD

	General	Strongly convex	Smooth
Average Regret	$\frac{1}{\sqrt{T}}$	$\frac{\log T}{T}$	$\frac{1}{\sqrt{T}}$

SVM in Machine Learning

Suppose $x_t \in \mathcal{X}$ is a feature vector and $y_t \in \{-1, 1\}$ is the label such that $\{(x_t, y_t)\}$ are **i.i.d samples** from a fixed (but unknown) distribution. Let there be T data points.

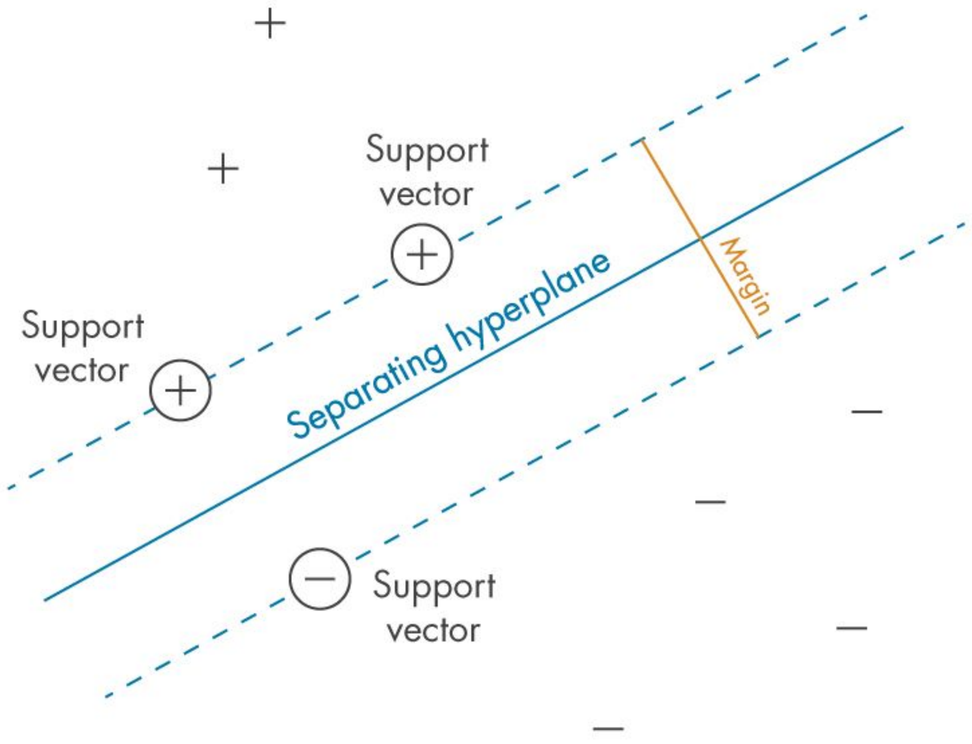
Goal: Find **a separating hyperplane** that minimizes the average hinge-loss: $h_T(w) = \frac{1}{T} \sum_{t=0}^{T-1} h(w; (x_t, y_t))$.



Support Vector Machines - Online Version

Suppose $x_t \in \mathcal{X}$ is a feature vector and $y_t \in \{-1, 1\}$ is the label that are revealed at every iteration (say by an **adversary**). Let there be T data points.

Goal: Find a sequence of hyperplanes that ensures sublinear regret against any set of T data points!



Hinge Loss

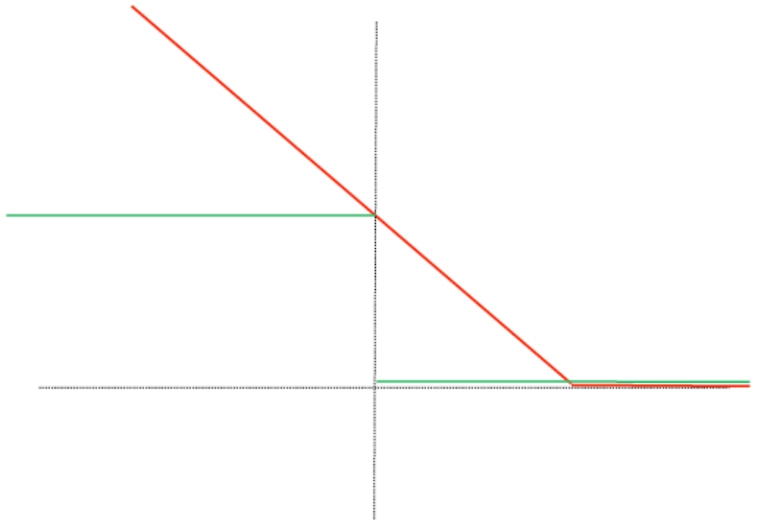


Figure 2.4: The hinge loss function versus the 0/1 loss function

SVM classifier on a data point (x, y) should satisfy, $y\big(\text{sign}(w^\top x)\big) \geq 1.$

Hinge loss: $h(w; (x, y)) = \max\{0, 1 - y(w^\top x)\}$

Mapping to Online Learning

- The game proceeds in rounds $t = 0,1,\ldots,T$.
- At each iteration t, Player plays $w_t \in \mathcal{X} := \{w : \|w\|_2 \leq B\}$ and gets $h_t(w_t) := h(w_t; (x_t, y_t))$ as feedback or loss.
- We will also assume $\|x_t\|_2 \leq G$, for all $t = 0,1,\ldots,T-1$.
- Player does not know future losses!
- The above sequence of loss functions are convex! But are they strongly convex or smooth?

Recall Online Gradient Descent

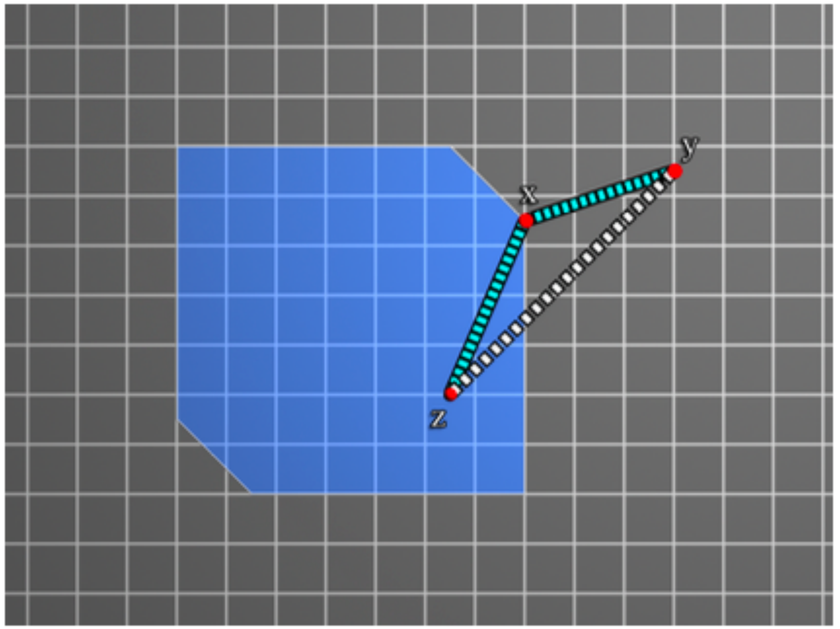
Initialize with $x_0 \in \mathcal{X}$

For $t = 0$ to $T - 1$ do,

$$y_{t+1} = x_t - \eta_t \nabla h_t(x_t)$$

$$x_{t+1} = \Pi_{\mathcal{X}}(y_{t+1})$$

Return x_T



OGD satisfies $R_T = o(T)$, with appropriate selection of $\{\eta_t\}$ and assuming uniformly bounded norm of gradients.

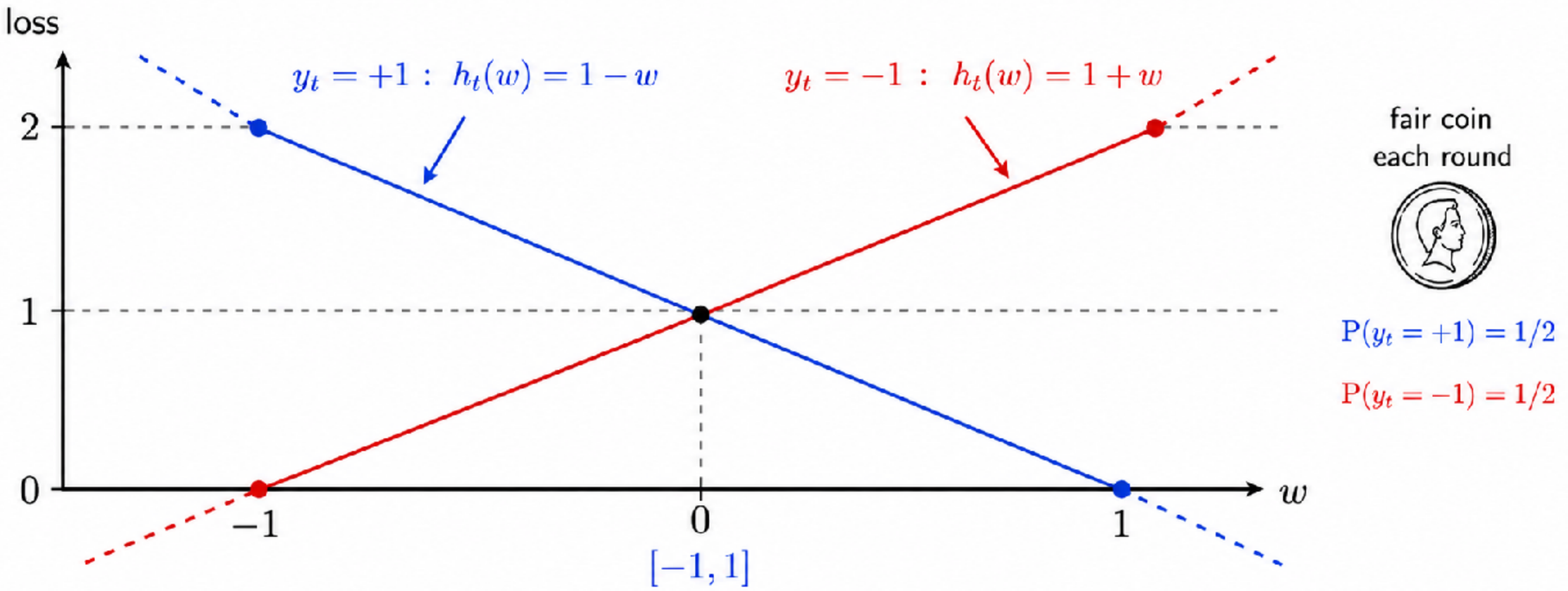
Online Gradient Descent for SVM

Under the stated assumptions, running OGD with step-size $\eta_t = \eta = B/(G\sqrt{T})$ for the online SVM problem produces a sequence of $\{w_t\}_{t=0}^{T-1}$ such that:

$$\sum_{t=0}^{T-1} h_t(w_t) - \min_{w \in \mathcal{X}} \sum_{t=0}^{T-1} h_t(w) \leq BG\sqrt{T}$$

Why?

Can we do better for SVM?

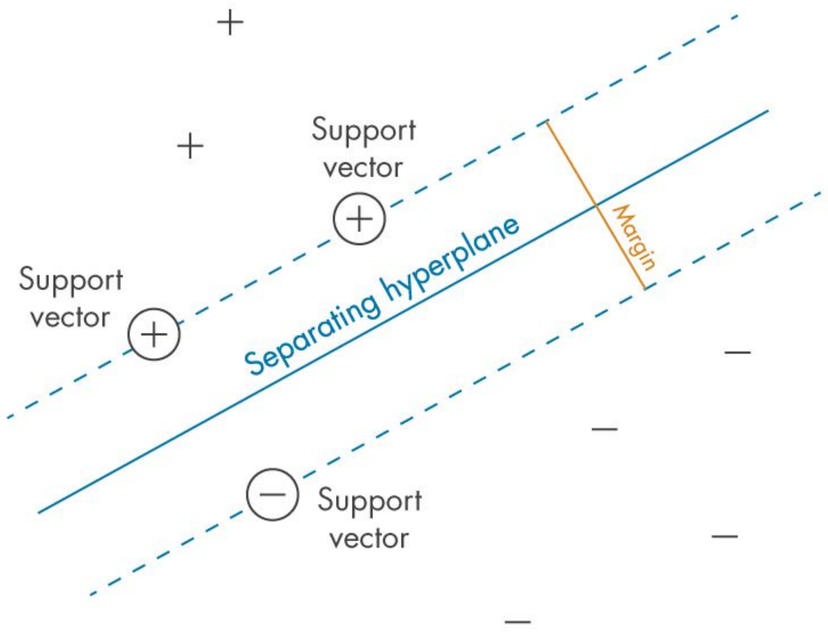


Suppose $x_t = 1$, for all t and $y_t \in \{-1,1\}$ is the label chosen by tossing a fair coin at every iteration. Let there be T data points. Then $\mathbb{E}[R(T)] = \Omega(\sqrt{T})$.

SVM in Machine Learning

Suppose $x_t \in \mathcal{X}$ is a feature vector and $y_t \in \{-1, 1\}$ is the label such that $\{(x_t, y_t)\}$ are **i.i.d samples** from a fixed (but unknown) distribution. Let there be T data points.

Goal: Find **a separating hyperplane** that minimizes the average hinge-loss: $h_T(w) = \frac{1}{T} \sum_{t=0}^{T-1} h(w; (x_t, y_t))$.



Stochastic Gradient Descent via OGD

If the (noisy) gradient at every step is **unbiased** and the **variance of the norm** of the gradient is bounded, then running OGD with these gradients and $\eta = B/(G\sqrt{T})$ leads to :

$$\mathbb{E}[h_T(\bar{w}_T)] \leq \min_{w \in \mathcal{X}} h_T(w) + 3GB/(2\sqrt{T}), \text{ where } \bar{w}_T = \sum_{t=0}^{T-1} w_t \,.$$

1. **Unbiased gradient:** $\nabla h(w) = \mathbb{E}_{(x,y)\sim\mathcal{D}}[\nabla h(w;(x,y))]$
2. **Gradient Norm Variance:** $\mathbb{E}_{(x,y)\sim\mathcal{D}}[\|\nabla h(w;(x,y))\|_2^2] \leq G^2$

See Theorem 3.4 in Hazan's book for the proof.

Summary

- Online learning is a powerful framework for optimization against the unknown.
- We saw OGD and the guarantees for online convex optimization.
- We looked at an online SVM example and why OGD is pretty much the best you can do.
- We saw that SGD can be seen as a special case of OGD. But in the upcoming classes we will treat it a separate algorithm.
- Next class: Distributed learning